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An Analogue of Montel's Theorem for Rational Functions of Best L_p- Approximation

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For sequences of rational functions, analytic in some domain, a theorem of Montel's type is proved. As an application, sequences of rational functions of best L_p-approximation with unbounded number of finite poles are considered.

INTRODUCTION

Given a domain B in the complex plane $\mathbf{C}$, let $\mathscr{F}$ be a family of functions, analytic and single valued in B ($\mathscr{F} \subset \mathscr{A}(B)$) and endowed with the sup-(uniform) norm on compact subsets of B.

By Montel's classical theorem about normal families of functions, if there are two points a and b, $a, b \in \mathbf{C}$, $a \neq b$, such that each function $f \in \mathscr{F}$ does not take the value of a in B and takes the value of b not more than at N different points in B, N—some fixed positive integer ($N \in \mathbf{N}$), then $\mathscr{F}$ is a normal family in B (cf. [Golusin]). We recall that a family $\tilde{\mathscr{F}}$ of functions analytic in some domain $\tilde{B}$ is normal, if from each infinite sequence $\{f_n\} \subset \tilde{\mathscr{F}}$ one can select a subsequence which converges uniformly (in the

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sup-norm) on each compact subset of $\tilde{B}$ either to an analytic in $\tilde{B}$ function, or to infinity.

Before we continue we recall that a compact set $E \subset \mathbf{C}$ is said to be regular, if its complement $E^c := \bar{\mathbf{C}} - E$ is connected and possesses a Green's function $G_E(z, \infty)$ with a pole at infinity such that for every $z_o \in \partial E$ $\lim_{z \to z_0, z \in E^c} G_E(z, \infty) = 0$. It is well known that if a compact set E is regular, then its Green's capacity $Cap\, E$ is positive (see, for instance, [Golusin], Chap. V). Given a number $\tau > 1$, we set $E_\tau := E \cup \{z,\ z \in \mathbf{C},\ G_E(z, \infty) < \log \tau\}$; $E_1 \equiv E$.

Let now $\mathscr{F}$ be a normal family in B. We observe that if some infinite sequence $\{f_n\} \subset \mathscr{F}$ converges uniformly on some regular compact subset of B, then it converges necessarily *locally uniformly inside* B (uniformly in the sup-norm on compact subsets of B).

Thus, a natural question arises: What happens if a sequence of analytic functions converges on a regular compact subset K of some domain B and does not take only one finite value in B? As it is known, in general only these conditions alone do not provide a uniform convergence inside the domain itself. This question appears to make sense for sequences of approximating rational functions. To make things clear, we recall the known result by Blatt *et al.* (cf. [BlSaSi]). For this purpose, we introduce, for a compact set E, $E \neq \emptyset$, a function f, continuous on $E (f \in C(E))$ and a nonnegative integer $n \in \mathbf{N}$, the notation $P_n(f, E)$; that is a polynomial of degree not exceeding n that best approximates f in the uniform norm on E.

THEOREM 1 [BlSaSi]. *Let E be a regular compact set in $\mathbf{C}$. Given a function f, $f \in \mathscr{A}(E^o) \cap C(E)$ and $f \not\equiv 0$ on each component of E, assume there is a domain U with nonempty intersection with E such that the number of zeros of $P_n(f, E)$ on each arbitrary compact set $K \subset U$ do not exceed $o(n)$ as $n \to \infty$.*

Then f admits an analytic continuation into the entire domain U. Furthermore, under the above conditions the sequence $\{P_n(f, E)\}$ converges locally uniformly inside E_ρ with $\rho = \inf\{\tau, U \subset E_\tau\}$ (and hence, $f \in \mathscr{A}(E_\rho)$).

This result is a consequence of the fundamental theorem of Jentzsch's type concerning the asymptotic distribution of zeros of polynomials of best uniform approximation. Another approach to the proof of Theorem 1, based on Leja's results, is given in [GiPl]. The method used in [BlSaSi] is extendable to rational functions $R_{n,m}(f, E)$ (n, the degree of the numerator; m, degree of the denominator) of best uniform approximation of f on E, with $n \to \infty$ and m fixed. An analogue of Theorem 1 is proved with an analytic continuation replaced by a meromorphic continuation with not more than m poles and rational functions converging to f locally uniformly inside U-{the poles of f} (the poles being counted with regard to their multiplicities). This result was established independently in [Kov1].

In these statements, an essential role is played by the fact that the denominator's degrees are fixed. It is an open problem whether a result of the Blatt–Saff–Simkani type holds for rational sequences $\mathscr{R}_{n,m_n}(f, E)$ of best uniform approximation when the degrees of the denominators $m_n \to \infty$.

In the present paper, sequences of rational functions of best approximation with unbounded numbers of the finite poles are considered.

Before presenting the results, we introduce some notations. For $n \in \mathbf{N}$, we denote by Π_n the collection of all polynomials of degree $\leqslant n$; for $n, m \in \mathbf{N}$, we set $R_{n,m}$ for all rational functions $\{r = p/q,\ p \in \Pi_n,\ q \in \Pi_m,\ q \not\equiv 0\}$. Given a compact set K and a function $h(z)$ defined on K, $v(h, K)$ will stand for the number of all zeros of $h(z)$ on K. For an infinite sequence $\{f_n\}$ and a domain B, the notation $\{f_n\} \subset \mathscr{N}(B)$ means that (a) $f_n \in \mathscr{A}(B)$ for all $n \in \mathbf{N}$ starting with a number $n_0 \in \mathbf{N}$ and (b) $v(f_n, K) = o(n)$ as $n \to \infty$ for every compact subset K of B. We will use the notations $\{f_n\} \subset \mathscr{U}_B(B)$ to express that $\{f_n\} \in \mathscr{A}(B)$ is uniformly bounded inside B (in the sup-norm on compact subsets), and $\{f_n\} \subset \mathscr{L}_f(B)$—that $\{f_n\}$ converges locally uniformly inside B to the function $f \in \mathscr{A}(B)$. In the sequel $\{f_n\} \subset \mathscr{L}(B)$ means that $\{f_n\}$ converges locally uniformly inside B. Finally, given a closed set $S \subset \mathbf{C}$, and a function f defined on S, we will write $f \in \mathscr{A}(S)$ to express that f is analytic in some neighborhood of S. Given a domain D, a closed set S, $S \subset D$ and function f, defined on S, we will write $f \in \mathscr{A}_S(D)$ if there is a function $F \in \mathscr{A}(D)$ such that $F \equiv f$ on S. Analogously, we define $f \in \mathscr{A}_G(D)$ with G being a subdomain in D.

STATEMENT OF THE RESULTS

The main result of the present paper is

THEOREM 2. *Let $\mathscr{S}$ be a regular continuum in $\mathbf{C}$ and B a domain, $B \supset \mathscr{S}$. Let $F := \{f_n\}_{n=1,2,\ldots}$ be a sequence of rational functions, $F \subset \mathscr{A}(B)$, with a total number of poles in $\overline{\mathbf{C}}$ of every f_n not exceeding n. Assume there is a function $f \in \mathscr{A}(S)$, $f \not\equiv 0$ on some regular subset of S and such that*

$$\limsup_{n \to \infty} \|f_n - f\|_{\mathscr{S}}^{1/n} < 1. \qquad (1)$$

If

$$v(f_n, K) = o(n) \qquad \text{as} \quad n \to \infty$$

for each compact subset K of B, then $F \subset \mathscr{L}(B)$; thus, $f \in \mathscr{A}_S(B)$.

In the present paper, we will apply Theorem 2 to rational functions of best L_p-approximation.

Let γ be a rectifiable curve in $\mathbf{C}$, p—a positive number and let f be a function of the class $L_p(\gamma)$ (this means that $|f|^p(t)$ is Lebesgue measurable on γ.) We adopt the notation $\|f\|_{L_p(\gamma)} := \{\int_\gamma |f(t)|^p |dt|\}^{1/p}$. For any pair (n, m) of nonnegative integers, let $\mathscr{R}_{n,m}^{(\gamma,p)}$ be a rational function of best L_p-approximation of f with respect to $\|\cdot\|_{L_p(\gamma)}$ in the class $R_{n,m}$; that is,

$$\|f - \mathscr{R}_{n,m}^{(\gamma,p)}\|_{L_p(\gamma)} := inf\ \|f - r\|_{L_p(\gamma)},$$

where the infimum is taken over $r \in R_{n,m}$.

THEOREM 3. *Let (n, m_n) be a sequence of nonnegative pairs, $m_n \leqslant n$, $m_n \leqslant m_{n+1}$, $n \to \infty$. Given a closed analytic curve Γ with an interior $\mathscr{G}$, a positive number p and a function f, $f \in L_p(\Gamma)$, assume*

$$\|\mathscr{R}_{n,m_n}^{(\Gamma,p)} - f\|_{L_p(\Gamma)} \to 0 \qquad as \qquad \to \infty. \tag{2}$$

Let U be a domain U, $U \supset \bar{\mathscr{G}}$ such that $\{\mathscr{R}_{n,m_n}^{(\Gamma,p)}\} \subset \mathscr{N}(U)$.
Then $f \in \mathscr{A}_\Gamma(U)$ and $\{\mathscr{R}_{n,m_n}^{(\Gamma,p)}\} \subset \mathscr{L}_f(U)$.

Remark 1. Given a weight function w, a.e. positive and integrable over the curve Γ, assume now $f \in L_{p,w}(\Gamma)$ (i.e., $|f|^p w$ is Lebesgue measurable on Γ). Set $\|f\|_{L_{p,w}(\Gamma)} := \{\int_\Gamma (|f|^p w)(t)\,|dt|\}^{1/p}$ and let $\mathscr{R}_{n,m_m}^{(\Gamma,p,w)} \in R_{n,m_n}$ be a rational function of best $L_{p,w}$-approximation of f on Γ. If $1/w^q$ is integrable over Γ for some positive number q, then Theorem 3 is extendable to $\{\mathscr{R}_{n,m_m}^{(\Gamma,p,w)}\}$.

As a particular case covered by Theorem 3 we point out rational functions of best L_p-approximation of functions of Hardy's class $H_p(:= H_p(\mathscr{T}), \mathscr{T}$—the unit circle), as well as L_p-weighted rational approximats of functions of the same class ($w > 0$ a.e. on $\mathscr{T}$.) In the space $H_p(:= H_p(\mathscr{T})$ condition (2) is necessarily fulfilled (see, for instance, [Timan, Chap. I].) Indeed, $\|\mathscr{P}_n - f\|_{L_p(\mathscr{T})} \to 0$ as $n \to \infty$ with $\mathscr{P}_n$ being a trigonometric polynomial of best L_p-approximation of degree n of f on $\mathscr{T}$; statement (2) results now from the minimality property of $\mathscr{R}_{n,m_n}^{(\mathscr{T},p)}$.

Set now $\Delta := [-1, 1]$ and let the weight function $w(x)$ be defined on Δ; $w > 0$ a.e. on Δ. Assume $f(x) \in L_{p,w}(\Delta)$.

Given a pair of nonnegative integers (n, m), let $\mathscr{R}_{n,m}^{(p,w)}$ be a rational function in the class $R_{n,m}$ of best $L_{p,w}$-approximation of f on Δ. Applying Theorem 2 we will establish

THEOREM 4. *Let $w(x)$ be a real-valued weight function, a.e. positive on Δ and integrable over Δ together with $w^{-q}(x)$ for a number $q > 0$. Assume that*

$f \in L_{p,w}(\Delta)$ is real-valued on Δ and let the sequence of pairs (n, m_n) be as in Theorem 3. If there is a domain U such that (a) *$U \supset \Delta$ and* (b) *$\{\mathcal{R}_{n,m_n}^{(p,w)}\} \subset \mathcal{N}(U)$, then $f \in \mathcal{A}_{\Delta}(U)$ and $\{\mathcal{R}_{n,m_n}^{(p,w)}\} \subset \mathcal{L}_f(U)$.*

As before, we observe that

$$\|\mathcal{R}_{n,m_n}^{(p,w)} - f\|_{L_{p,w}(\Delta)} \to 0 \qquad as \quad n \to \infty. \tag{3}$$

BACKGROUND

The first result exploring the connection between zeros and analytic continuability is the known Bernstein's theorem;

THEOREM 5 [Be]. *Let the function $f \in C(\Delta)$ be real-valued on the interval Δ. Assume that there is an ellipse $\mathcal{E}$ with foci at ± 1 such that all polynomials $P_n(f, \Delta)$ are, starting with a number n_o, nowhere zero in its interior $\mathcal{E}^o$.*

Then $f \in \mathcal{A}_{\Delta}(\mathcal{E}^o))$ and $\{P_n(f, \Delta)\} \subset \mathcal{L}_f(\mathcal{E}^o)$.

In the same paper, Bernstein pointed out that Theorem 5 holds only for polynomials of best approximation.

Another result of Montel's type is:

THEOREM 6 [BGrM]. *Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a power series with a positive radius of convergence, let $\{\pi_n\}_{n=1,2,\ldots}$ be the main diagonal sequence in the Padé table associated with f and U be a disk centered at $z = 0$ such that $\{\pi_n\} \subset \mathcal{N}(U)$ for every n starting with some n_0.*

Then $f \in \mathcal{A}(U)$ and $\{\pi_n\} \subset \mathcal{L}_f(U)$.

Originally, Theorem 6 was established under the assumption that $\{\pi_n\} \subset \mathcal{U}_B(U)$. It is easy to show that $\{\pi_n\} \subset \mathcal{N}(U)$ leads to $\{\pi_n\} \subset \mathcal{U}_B(U)$. Later, in 1982, the statement of Theorem 6 was established by A. A. Gonchar under essentially weaker conditions, involving only $\{\pi_n\} \in \mathcal{A}(D)$, $n = 1, 2, \ldots, 0 \in D$ with D representing a large class of sets in $\mathbf{C}$ (cf. [Gon1]).

Further results of Montel's type were obtained in [GrthSaff, Kov2, Kov3].

PRELIMINARIES

Let $f, g \in L_p[a, b]$. We first remind of the basic fact that

$$\|f + g\|_{L_p[a,b]}^p \leqslant C(\|f\|_{L_p[a,b]}^p + \|g\|_{L_p[a,b]}^p)$$

with $C = \max[1, 2^{p-1}]; -\infty < a < b < \infty$ (cf. [Walsh1]). If $p \geqslant 1$, then the Minkowski inequality is valid, i.e.,

$$\|f+g\|_{L_p[a,b]} \leqslant \|f\|_{L_p[a,b]} + \|g\|_{L_p[a,b]}.$$

Given a set e of positive Lebesgue measure (we write $m(e) > 0$), let the functions f_n, $n = 1, 2, \ldots$ be defined on e. The sequence $\{f_n\}$ is said to converge in measure on e, if for every positive ε and δ there holds $m\{z, z \in e, |f_n(z) - f_m(z)| \geqslant \varepsilon\} < \delta$ for all numbers n, m large enough (cf. [Natanson]). By a theorem of Natanson, if some sequence $\{f_n\}$ converges to a function f in $L_p[a, b]$, then it converges in measure on $[a, b]$, too (cf. [Natanson]).

The function f is said to belong to Hardy's class H_p (see, for instance, [Privalov]) if (a) $f \in \mathscr{A}(T)$, $T :=$ the unit disk and (b) $\sup_{\rho \to 1} \int_0^{2\pi} |f(\rho \exp(i\tau)|^p \, d\tau$ is bounded. (If $p = \infty$, then $\sup_{\rho \to 1} |f(\rho \exp(i\tau)|$ should be bounded, respectively.)

If $f \in H_p$, then the nontangential limits $\lim_{z' \to z, z' \in T} f(z')$ exist for almost all z, $z \in \mathscr{T}$ ($\mathscr{T} :=$ the unit circle; cf. [Privalov]). One can define $f(\exp(i\tau))$, $\tau \in [0, 2\pi]$ as the limit of $f(\rho \exp(i\tau))$ as $\rho \to 1$. It is customary to write $f(\exp(i\tau))$ instead of $\lim_{\rho \to 1} f(\rho \exp(i\tau))$. Recall that the nontangential limit function $f(\exp(i\tau)) \in L_p(\mathscr{T})$ (cf. [Privalov]). If $f, g \in H_p$ and $f = g$ for $z \in E$, with E being a subset of $\mathscr{T}$ of positive measure, then $f \equiv g$ (Privalov's uniqueness theorem for H_p (cf. [Privalov].)) We recall that the uniqueness theorem preserves its validity under the same condition (namely, $f = g$ on a subset of $\mathscr{T}$ of positive measure) for functions analytic and single valued in T, also—the theorem of Privalov–Luzin (cf. [Privalov]). Further, according to Ostrowski–Khinchine's theorem, if some sequence $\{f_n\}$ with $f_n \in H_p$ and $\|f_n\|_{L_p(\mathscr{T})} \leqslant C_1$ for all n converges on $\mathscr{T}$ in measure to some function f, then $\{f_n\} \subset \mathscr{L}_F(T)$ with a limit function F coinciding with f a.e. on $\mathscr{T}$ and being an element of H_p.

In the sequel, C_n, $n = 1, 2, \ldots$, denote positive constants which do not depend on the integer n and are different at different occurrences.

The proofs will be preceded by a few lemmas.

Let $F := \{F_n\}_{n=1,2,\ldots}$ be functions locally single-valued and analytic in some domain B except perhaps for branch points, and let each $|F_n|$ starting with some number n_0 be single-valued there. We say that some harmonic function v is *a harmonic majorant* for F in B, if for every compact subset M of B the inequality

$$\limsup_{n \to \infty} \|F_n\|_M \leqslant \exp \|v\|_M$$

is valid.

LEMMA 1 [Walsh2]. *Let B be a domain in $\mathbf{C}$, $F := \{F_n\}_{n=1}^{\infty}$ be a sequence as above and let v be a harmonic majorant for F in B.*

If there is a regular compact set M, $M \subset B$, where a strict inequality holds, i.e., if

$$\limsup_{n \to \infty} \|F_n\|_M < \exp \|v\|_M,$$

then a strict inequality holds on every compact subset of B.

Of importance for the coming considerations is

LEMMA 2. *Let γ be a closed analytic curve in $\mathbf{C}$ and p a positive number. Denote by D the finite domain bounded by γ. Let $g \in \mathscr{A}(D) \cap C(\bar{D})$.*

Then for each compact subset K of D there exists a constant $C_1 = C_1(K, p)$ such that

$$\|g\|_K \leqslant C_1 \|g\|_{L_{p(\gamma)}}.$$

The case when γ coincides with the unit circle $\mathscr{T}$ was considered by Walsh (cf. [Walsh1, Chap. V]). The proof for the general case proceeds, after mapping conformally $\bar{D}$ onto $\bar{T}$, along the same line of reasoning, so we may omit it.

LEMMA 3 [Gon2]. *Let E be a regular compact set in $\mathbf{C}$ and D a domain, $D \supset E$. Given a sequence $\{\mathscr{R}_n\}$ of rational functions each with a total number of poles in $\overline{\mathbf{C}} \leqslant n$, assume $R_n \in \mathscr{A}(D)$ for every n.*

Then for every compact set $K \subset D - E$ there exists a constant $\lambda_1(K) := \lambda_1 > 1$ such that for every $n \in \mathbf{N}$ the inequality

$$\|\mathscr{R}_n\|_K \leqslant \lambda_1^n \|\mathscr{R}_n\|_E$$

holds.

The constant $\lambda_1(K)$ is given by $\lambda_1(K) = \sup_{z' \in K, z'' \in D^c} G_E(z', z'')$.

A similar result holds for $L_{p,w}$-norms, too. To be precise, we present

LEMMA 4. *Let E be a regular continuum and $w(x)$—a weight function, a.e. positive and integrable over ∂E together with $w(x)^{-q}$ for some positive q.*

Then there are for every $p > 0$ and for every compact set K, $K \subset D - E$, a positive constants $C_2 = C_2(K, w, p)$ and $\lambda_2(K) = \lambda_2 > 1$, such that

$$\|\mathscr{R}_n\|_K \leqslant C_2 \lambda_2^n \|\mathscr{R}_n\|_{L_{p,w(\partial E)}}.$$

For the particular case, when ∂E is a closed analytic curve, Lemma 3 was proved by Walsh (cf. [Walsh1, Chap.V]). The proof of the form presented here follows the main idea of Walsh.

Analyzing the proofs of Lemma 3 and Lemma 4 we arrive at

Remark 2. Let $\{K_n\}$ be a sequence of compact sets, $\ldots \supset K_n \supset K_{n+1} \supset \ldots$, $E = \cap K_i$. Then $\lambda_i(K_n, p) \to 1^+$ as $n \to \infty$, $i = 1, 2$.

In the forthcoming proofs we shall need an inequality of Nikolski-type that estimates from above the uniform norm of a polynomial on the interval Δ by its L_p-norm. Before, we recall some known definitions and facts from [KrSwet].

Let w be a weight function, a.e. positive on Δ and such that $\int_{-1}^{1} w(x)\, dx = 1$. As usual, μ stands for the equilibrium measure on Δ ($\mu := \frac{1}{\pi} \int_\Delta (dx/\sqrt{1-x^2})$). Given a positive number ε, set $\phi(w, \varepsilon) := \inf\{\int_A w(x)\, dx : A \subset [-1, 1], \mu(A) \geqslant \varepsilon\}$ and let $\varepsilon_n(w)$ be a solution of the equation $\phi(w, \varepsilon) = \exp(-n\varepsilon)$. As it is known (see [KrSwet]), $\varepsilon_n(w)$ is unique and $\varepsilon_n(w) \to 0$ as $n \to \infty$.

LEMMA 5 [KrSwet]. *Let w be a weight function as above. Then for every $p > 0$ and for every polynomial $p_n \in \Pi_n$, the inequality*

$$\|p_n\|_\Delta \leqslant \exp(cn\varepsilon_n(w))\, \|p_n\|_{L_p(\Delta)}$$

holds, where the positive constant $c > 0$ depends only on p and w.

PROOFS

Proof of Theorem 2. Under the conditions of the theorem and by means of Lemma 3, Remark 2, there is a compact set $\tilde{S}$, $\tilde{S}^0 \neq \varnothing$, $S \subset \tilde{S} \subset B$ such that

$$\limsup_{n \to \infty} \|f_{n+1} - f_n\|_{\tilde{S}}^{1/n} < 1. \tag{4}$$

Hence, f_n converge as $n \to \infty$ uniformly on $\tilde{S}$ to some function $g \in \mathscr{A}(\tilde{S})$. In view of the conditions of the theorem, $g \not\equiv 0$ on $\tilde{S}$ and $f \equiv g$ on $\tilde{S}$. Let $\tilde{S}'$ be a regular subset of $\tilde{S}$ of nonempty interior such that $g \neq 0$ on $\tilde{S}'$. By means of the classical Hurwitz's theorem,

$$|f_n|^{1/n} \to 1, \qquad \text{as} \quad n \to \infty \tag{5}$$

uniformly on $\tilde{S}'$.

Select now a simply connected domain W, satisfying $\tilde{S} \subset W \subset B$. For every n large enough, say $n > n_1$, let $\pi_n(z) := \prod_{i=1}^{k_n}(z - \chi_{i,n})$ be the monic polynomial with zeros at all zeros of f_n on $\bar{W}$. (If $k_n = 0$, then $\pi_n(z) \equiv 1$.) In view of the hypothesis of the theorem,

$$k_n := o(n) \qquad \text{as} \quad n \to \infty. \tag{6}$$

Observe that $\pi_n(z) \neq 0$ for $z \in \tilde{S}'$, as well as for z in some neighborhood of $\tilde{S}'$. Therefore, by (5) and (6),

$$|\pi_n(z)|^{1/n} \to 1 \qquad \text{as} \quad n \to \infty \tag{7}$$

uniformly on $\tilde{S}'$.

On fixing an arbitrary point b in $\tilde{S}'$, we introduce into consideration the sequence $\chi := \{\chi_n\}_{n=n_1}^{\infty}$ with $\chi_n := \{\pi_n^{-1} \cdot f_n\}^{1/n}$ and every χ_n being that regular branch in W for which $|arg\ \chi_n(b)| \leqslant 1/n$. Obviously, $\chi \subset \mathscr{A}(W)$.

We now claim that

$$\chi_n(z) \to 1 \qquad \text{as} \quad n \to \infty \tag{8}$$

locally uniformly inside W.

Indeed, regarding (5) and (7), as well as the choice of the regular branches, we see that (8) is valid uniformly on $\tilde{S}'$. Hence, by Lemma 3, $\{\chi_n\} \subset \mathscr{U}_B(W)$. Statement (8) follows now after applying subsequently the compactness principle and Vitali's theorem.

From (8) and (6) we get

$$\limsup_{n \to \infty} \|f_n\|_K^{1/n} \leqslant 1$$

and

$$\limsup_{n \to \infty} \|f_{n+1} - f_n\|_K^{1/n} \leqslant 1$$

with K being any compact subset of W. By (1), the above inequality is strict on S. Hence, owing to Lemma 1, a strict inequality holds on each compact subset of W. Thus, $\{f_n\} \subset \mathscr{L}(W)$, and, regarding (1), $\{f_n\} \in \mathscr{L}_f(W)$. On letting W tend to B, we arrive at the statement of Theorem 2. Q.E.D.

Proof of Theorem 3. Set $\mathscr{R}_{n,m_n}^{(\Gamma,p)} := \mathscr{R}_n$, $n = 1, 2, \ldots$.

Fixing an arbitrary point z_o in $\mathscr{G}$, let ϕ be the unique univalent function which maps $\mathscr{G}$ on $\{u, |u| < 1\}$ in a way that $\phi(z_o) = 0$ and $\phi'(z_o) > 0$. The function ϕ maps Γ onto $\{u, |u| = 1\}$ in a one-to-one way. Further, both $\phi, \phi' \in \mathscr{A}(\mathscr{G}) \cap C(\bar{\mathscr{G}})$. We remark that (in the case being considered) ϕ' is nonzero in $\bar{\mathscr{G}}$ (theorems of Caratheodory and of Lindelöf, cf. [Golusin]).

Let $\psi(u)$ be the inverse of $\phi(z)$; we recall that $\psi' \in \mathscr{A}(T) \cap C(\bar{T})$ and $\psi'(u) \neq 0$ for $|u| \leqslant 1$. Let $(\psi')^{1/p}$ be that regular branch for which $(\psi'(0))^{1/p} > 0$.

Set $r_n := (\mathscr{R}_n \circ \psi)(\psi')^{1/p}$; apparently, $r_n \in H_p$, $n = 1, 2, \ldots$.

By means of Natanson's theorem, $\{r_n\}$ converges in measure on $\mathscr{T}$ as $n \to \infty$. With $\tilde{\Psi}$ being the limit function, we note that $\tilde{\Psi}(u) = (f \circ \psi(u))\, \psi'(u)^{1/p}$ a.e. on $\mathscr{T}$.

From (2), we get

$$\|r_n\|_{L_p(\mathscr{T})} \leqslant C_3 \tag{9}$$

for n large enough, say $n \geqslant n_2$. Consequently, regarding the Ostrowski–Khinchine theorem, $\{r_n\} \subset \mathscr{L}(T)$ and the limit function r, being an element of H_p, coincides with $\tilde{\Psi}$ a.e. on $\mathscr{T}$; hence $r(u) = (f \circ \psi)(u)\, \psi'(u)^{1/p}$ a.e. on $\mathscr{T}$. Recall that $r(\exp i\tau)$ should be regarded as $\lim_{\rho \to 1} r(\rho \exp i\tau)$.

Setting $r \circ \phi := \Psi$, we get

$$\lim_{n \to \infty} \mathscr{R}_n(z) = \Psi \tag{10}$$

locally uniformly inside $\mathscr{G}$; $\Psi \in \mathscr{A}(\mathscr{G})$. We remark that the nontangential limits Ψ exist for almost all $z \in \Gamma$ and $\Psi = f$ a.e. there.

Note that by virtue of Privalov's uniqueness theorem, $\Psi \not\equiv 0$ in $\mathscr{G}$.

In the same way as in the proof of Theorem 2, we introduce the simply connected domain W such that $\bar{\mathscr{G}} \subset W \subset U$. Using the same argumentation, we see that

$$\lim_{n \to \infty} \|\mathscr{R}_n\|_K^{1/n} \leqslant 1 \tag{11}$$

for each compact set K in W .

We are now going to prove that

$$\limsup_{n \to \infty} \|f - \mathscr{R}_n\|_{L_p(\Gamma)}^{1/n} < 1. \tag{12}$$

For this purpose, we introduce, for $\rho > 1$, the level curve $\Gamma_\rho := \{z, G_{\mathscr{G}}(z, \infty) = \ln \rho\}$.

Select a number μ, $1 < \mu < \sup\{\rho, \Gamma_\rho \subset W\}$. Let $w = \{w_n\}_{n=1,2,\ldots}$ be a sequence of monic polynomials, nonzero in $\mathscr{G}^c$ and satisfying, for every $\rho > 1$

$$\lim_{n \to \infty} \left\| \frac{w_n(u)}{w_n(v)} \right\|_{u \in \Gamma,\, v \in \Gamma_\rho}^{1/n} = \frac{1}{\rho}.$$

For each n, let W_n be the polynomial of degree n which interpolates the rational function $\mathscr{R}_{n+1}(z)$ at all zeros of $w_{n+1}(z)$. The application of the Hermite–Lagrange interpolation formula yields for each $z \in \Gamma$

$$\mathscr{R}_{n+1}(z) - W_n(z) = \frac{1}{2\pi i} \int_{\Gamma_\mu} \frac{w_{n+1}(z)}{w_{n+1}(t)} \frac{\mathscr{R}_{n+1}(t)}{t-z} \, dt.$$

Select now a positive number Θ_1 such that $\exp \Theta_1 < \mu$. For all n sufficiently large, say $n > n_3$, we may write, after taking into account (11)

$$\|\mathscr{R}_{n+1} - W_n\|_{\bar{\mathscr{G}}} \leqslant C_4 (\exp \Theta_1 / \mu)^n. \tag{13}$$

On the other hand, we have an obvious inequality

$$\|\mathscr{R}_{n+1} - W_n\|_{L_p(\Gamma)} \leqslant C_5 \, \|\mathscr{R}_{n+1} - W_n\|_{\bar{\mathscr{G}}}. \tag{14}$$

Consider first the case when $p < 1$. By the minimality property, we have

$$\begin{aligned} \|f - \mathscr{R}_n\|^p_{L_p(\Gamma)} \leqslant \|f - W_n\|^p_{L_p(\Gamma)} &\leqslant \|f - \mathscr{R}_{n+1}\|^p_{L_p(\Gamma)} \\ &\quad + \|\mathscr{R}_{n+1} - W_n\|^p_{L_p(\Gamma)}. \end{aligned}$$

After taking into account (13) and (14), we obtain

$$\|f - \mathscr{R}_n\|^p_{L_p(\Gamma)} - \|f - \mathscr{R}_{n+1}\|^p_{L_p(\Gamma)} \leqslant C_6 \{\exp \Theta_1 / \mu\}^{pn}. \tag{15'}$$

For $p \geqslant 1$, after handling $\|f - \mathscr{R}_n\|_{L_p(\Gamma)}$ similarly and applying the Minkowski inequality, we get

$$\|f - \mathscr{R}_n\|_{L_p(\Gamma)} - \|f - \mathscr{R}_{n+1}\|_{L_p(\Gamma)} \leqslant C_7 \{\exp \Theta_1 / \mu\}^n. \tag{15''}$$

In view of the of the conditions of the theorem, for every n the inequality $\|f - \mathscr{R}_n\|_{L_p(\Gamma)} \geqslant \|f - \mathscr{R}_{n+1}\|_{L_p(\Gamma)}$ holds. With this remark, (12) follows now from inequalities (15) and (2), after passing to the limit.

Let now E be a regular continuum in $\mathscr{G}$. The application of Lemma 2 leads, thanks to (12), to the inequality

$$\limsup_{n \to \infty} \|\mathscr{R}_{n+1} - \mathscr{R}_n\|_E^{1/n} < 1.$$

Hence, regarding (10), we get

$$\limsup_{n \to \infty} \|\mathscr{R}_n - \Psi\|_E^{1/n} < 1.$$

Thus, all conditions of Theorem 2 are fulfilled with respect to the sequence $\{\mathscr{R}_n\}$. By this, Theorem 3 is proved. Q.E.D.

Proof of Remark 1. Recall that w is nonnegative on Γ and integrable together with w^{-q} for some $q>0$. Following [Walsh1, Chap. V], we have, by the Hölder inequality

$$\int_\Gamma |f(t)-\mathscr{R}_{n,m_n}^{(\Gamma,p,w)}(t)|^{pq/(1+q)}\,|dt|$$

$$\leqslant \left(\int_\Gamma \frac{1}{w(t)^q}\,|dt|\right)^{1/(1+q)} \left(\int_\Gamma w(t)\,|f(t)-\mathscr{R}_{n,m_n}^{(\Gamma,p,w)}(t)|^p\,|dt|\right)^{q/(1+q)}.$$

Joining (12), we are capable of proving analogues of Lemmas 2. and 3. for weighted L_p-approximation. With these notations, the proof in question follows the main idea of the proof of Theorem 3. Q.E.D.

Proof of Theorem 4. Set, as before, $\mathscr{R}_n := \mathscr{R}_{n,m_n}^{(p,w)}$. We presume f not to vanish identically on some subinterval Δ^* of Δ of positive length. Additionally, we assume that $\int_{-1}^1 w(x)\,dx=1$ and that U is a bounded domain.

Thanks to (3), we have

$$\|\mathscr{R}_n\|_{L_{p,w}(\Delta)} \leqslant C_8 \tag{9'}$$

with C_6 being an appropriate positive constant and $n\geqslant n_4$. We remark that (9′) preserves its validity for any subinterval of Δ'. Taking into account (9′), (3) and Remark 2, we obtain

$$\lim_{n\to\infty} \|\mathscr{R}_n\|_{\Delta'}^{1/n}=1 \tag{16}$$

for any regular subinterval Δ' of Δ.

Set $\mathscr{R}_n=P_n/Q_n$, $Q_n(z)=\prod(z-\eta'_{n,i})\prod(1-z/\eta''_{n,i})$, where η'_n and η''_n are those zeros of Q_n which are situated on the disk $D:=\{z, |z|\leqslant 2\,diam(U)\}$ and outside, respectively.

Let now W, V be simply connected domains such that $\Delta\subset\bar{W}\subset V\subset U$. As above, let $\pi_n, n=n_4, \ldots$ be the monic polynomial of degree $k_n=\nu(\mathscr{R}_n,\bar{V})$, the zeros of which coincide with all zeros of $\mathscr{R}_n$ on $\bar{V}$. Recall that

$$k_n=o(n) \qquad \text{as} \quad n\to\infty.$$

Hence, for every regular compact set K in W,

$$\limsup_{n\to\infty} \|\pi_n\|_K^{1/n}\leqslant 1.$$

On the other hand, we have (cf. [Golusin, Chap.V])

$$\liminf_{n\to\infty} \|\pi_n\|_K^{1/k_n}\geqslant Cap\,K. \tag{17}$$

Combining both inequalities, we get, for every compact subset K of W, that

$$\lim_{n\to\infty} \|\pi_n\|_K^{1/n} \to 1. \tag{18}$$

For any n, $n \geqslant n_4$, select a number a_n, $a_n \in \Delta$ such that $\mathscr{R}_n(a_n) \neq 0$ and introduce $\chi_n := \{\mathscr{R}_n \pi_n^{-1}\}^{1/n}$ with $|arg\, \chi_n(a_n)| \leqslant 1/n$. Apparently, the rational functions χ_n do not vanish on $\bar{W}$ and $\{\chi_n\} \in \mathscr{A}(W)$.

On writing $P_n = \pi_n p_n$, we observe that $\chi_n := (\frac{p_n}{Q_n})^{1/n}$. We claim that

$$\chi_n \to 1 \qquad \text{as} \quad n \to \infty \tag{19}$$

locally uniformly inside W.

First, we show that $\{\chi_n\} \in \mathscr{U}_B(W)$.

Indeed,

$$\|\mathscr{R}_n\|_{L_p(\Delta)} = \frac{1}{|Q_n(\xi_n)|} \|P_n\|_{L_p(\Delta)}$$

for an appropriate point ξ_n, $\xi_n \in \Delta$; this equality implies together with (9′)

$$\|P_n\|_{L_p(\Delta)} \leqslant C_8 \, \|Q_n\|_\Delta .$$

By means of Lemma 5

$$\|P_n\|_\Delta \leqslant C_9^n \, \|Q_n\|_\Delta, \qquad n \geqslant n_5,$$

where C_9 stands for $\exp c$ and $n \geqslant n_5$ is sufficiently large such that $\varepsilon_n(w) < 1$.

Select now a number R, $R > 1$ in the way that the interior of the ellipse $\mathscr{E}_R := \{z = x + iy, (2Rx/(R^2+1))^2 + (2Ry/(R^2-1))^2 = 1\}$ contains $\bar{U}$. Estimating now $\|P_n\|_{\mathscr{E}_R}$ by the lemma of Bernstein–Walsh, we come to

$$\|P_n\|_{\mathscr{E}_R} \leqslant \|P_n\|_\Delta \, R^n \leqslant \|Q_n\|_{(\Delta)} \, C_9^n R^n,$$

which implies

$$\|P_n\|_{\mathscr{E}_R} \leqslant C_{10}^n R^n.$$

From here we get

$$\|p_n\|_{\mathscr{E}_R} \leqslant \frac{C_{10}^n R^n}{C_{11}^{k_n}}$$

with $C_{11} := \min\{|\pi_n(z)|, z \in \mathscr{E}_R\}$. Recalling that $k_n = o(n)$ as $n \to \infty$, we arrive at

$$\|p_n\|_{\mathscr{E}_R} \leqslant C_{12}^n \tag{20}$$

with C_{12} depending on R and W but not on n. Given a compact subset K of W, we have

$$\begin{aligned}\|\chi_n\|_K &\leqslant \{\|p_n\|_K / min\{|Q_n(z)|, z \in K\}\}^{1/n} \\ &\leqslant \{\|p_n\|_{\mathscr{E}_R} / min\{|Q_n(z)|, z \in W\}\}^{1/n} \leqslant C_{13}.\end{aligned}$$

Thus, $\{\chi_n\} \in \mathscr{U}_B(W)$.

Let now Δ' be an arbitrary regular subinterval of Δ. Thanks to

$$|\mathscr{R}_n(z)|^{1/n} = |\pi_n(z)|^{1/n}\, |\chi_n(z)| \tag{21}$$

and with regard to (16) and (18), we may write

$$\liminf_{n \to \infty} \|\chi_n\|_{\Delta'} \geqslant 1.$$

Hence, there exists a sequence $\{\kappa_n\}$, $\kappa_n \in \Delta'$ such that $\liminf_{n \to \infty} |\chi_n(\kappa_n)| \geqslant 1$. On the other hand, regarding again (21) and taking into account (17), we obtain that

$$\limsup_{n \to \infty} |\chi_n(\tau_n)| \leqslant 1$$

for an appropriate sequence $\{\tau_n\}$, $\tau_n \in \Delta'$, $|\pi_n(\tau_n)| = \|\pi_n\|_{\Delta'}$.

Let now $\tilde{X}$ be any limit function of the sequence $\{\chi_n\}$; that is: $\tilde{X} = \lim_{n \in \Lambda} \chi_n$ locally uniformly inside W for some infinite sequence Λ. Keeping in mind both last relations, we see that $|\tilde{X}|$ takes the value of unity at least one time on each arbitrary regular subinterval Δ' of Δ^*. Hence, $|\tilde{X}| \equiv 1$ on Δ^*. Statement (19) results immediately from the symmetry of the functions $\mathscr{R}_n$, $n = 1, 2, \ldots$ with respect to the real axes, after keeping track of the choice of the regular branches χ_n.

Thanks to the arbitrariness of W, the convergence (19) takes place everywhere inside the entire domain U (on compact subsets).

Coming back to the functions $\mathscr{R}_n$, we get, regarding (18), that

$$\limsup \|\mathscr{R}_n\|_K^{1/n} \leqslant 1 \qquad \text{as} \quad n \to \infty$$

for every compact subset K in U.

In the same way, as in the previous proofs, we show that

$$\limsup_{n\to\infty} \|f-\mathscr{R}_n\|_{\Delta}^{1/n} < 1.$$

Hence,

$$\limsup_{n\to\infty} \|\mathscr{R}_{n+1}-\mathscr{R}_n\|_{L_{p,w}(\Delta)}^{1/n} < 1$$

and

$$\limsup_{n\to\infty} \|\mathscr{R}_{n+1}-\mathscr{R}_n\|_{E}^{1/n} < 1$$

for an appropriate regular compact set E of nonempty interior with $\Delta \subset E \subset U$. Thus, $\{\mathscr{R}_n\}$ converges uniformly on E to a function, say $\Phi(z)$, where $\Phi(z) \in \mathscr{A}(E)$, $\Phi(z) \equiv f$ a.e. on Δ and $\limsup_{n\to\infty} \|f-\mathscr{R}_n\|_E^{1/n} < 1$. By this, all conditions of Theorem 2 are fulfilled with respect to the sequence $\{\mathscr{R}_{n,m_n}\}$ and Theorem 2 is applicable. Q.E.D.

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